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Some remarks on $\mathfrak{g}$-invariant Fedosov star products and quantum momentum mappings

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Abstract

In these notes we consider a slightly generalized Fedosov star product $*$ on a symplectic manifold (M, ω), emanating from the fibrewise Weyl product $\circ$ and the triple (∇, Ω, s) consisting of a symplectic torsion free connection ∇ on M, a formal series $\Omega \in \nu Z^2_{\mathrm{dR}}(M)[[\nu]]$ of closed two-forms on M, and a certain formal series s of symmetric contravariant tensor fields on M. We prove necessary and sufficient conditions for certain classical symmetries to become symmetries of the star product, only sufficient conditions having been published in special cases when this letter was written (note, however, the different proofs in [S. Gutt, J. Rawnsley, Natural star products on symplectic manifolds and quantum moment maps, 2003. math.SG/0304498 v1]). For a given symplectic vector field X on M, it is well known that $\mathcal{L}_X\Omega = [\mathcal{L}_X, \nabla]\ (=\mathcal{L}_X s) = 0$ is a sufficient condition for the Lie derivative $\mathcal{L}_X$ to be a derivation of $*$. We prove that these conditions are in fact necessary ones, also providing a very simple proof for their being sufficient. Moreover, we prove a criterion that has first been presented by Gutt [S. Gutt, Star products and group actions, Contribution to the Bayrischzell Workshop, April 26–29, 2002] (see also [S. Gutt, J. Rawnsley, Natural star products on symplectic manifolds and quantum moment maps, 2003. math.SG/0304498 v1] for a different proof) and which specifies a necessary and sufficient condition for $\mathcal{L}_X$ to be a quasi-inner derivation. The statement that this condition is a sufficient one dates back to Kravchenko [O. Kravchenko, Compos. Math. 123 (2000) 131]. Applying our results, we find necessary and sufficient criteria for a Fedosov star product to be $\mathfrak{g}$-invariant and to admit a quantum Hamiltonian. Finally, supposing the existence of a quantum Hamiltonian, we present a cohomological condition on Ω that is equivalent to the existence of a quantum momentum mapping. In particular, our results show that the existence of a classical momentum mapping in general does

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not imply the existence of a quantum momentum mapping and thus give a negative answer to Xu's question posed in [P. Xu, Commun. Math. Phys. 197 (1998) 167].

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1. Introduction

Already very early in the development of deformation quantization (star products) as introduced in [2] by Bayen et al., various notions of invariance of star products with respect to Lie group and Lie algebra actions were introduced and discussed by Arnal et al. [1]. General invariant star products on symplectic manifolds have been classified up to equivalence by Bertelson et al. [3], and even stronger classification results have meanwhile been obtained for star products of Wick type on Kähler manifolds by the authors [14].

An important structure in deformation quantization, which naturally generalizes the momentum mapping in Hamiltonian mechanics, is that of a quantum momentum mapping for $\mathfrak{g}$-invariant star products as defined and studied in detail by Xu [18]. This notion has proved to be essential for the formulation of the analogue of the Marsden–Weinstein reduction in deformation quantization as it was studied in [9] as well as for the example of reduction of star products for $\mathbb{C}P^n$ [5,17] and the application of the BRST quantization in deformation quantization [6].

In [18], Xu raised the question whether the existence of a classical momentum mapping guarantees the existence of a quantum momentum mapping in his sense. Recently, the question of existence of a quantum momentum mapping for the usual Fedosov star products has been taken up by Hamachi [12], who has given a condition for its existence in terms of parts of the Fedosov derivation used to define the star products assumed to be invariant with respect to a symplectic Lie Group action on M. A systematic treatment of these questions in the case of actions of a Lie group was also presented by Gutt [10].

In the present paper, we generalize these results in two aspects. First, we drop the assumption of invariance of the (generalized) Fedosov star products with respect to a Lie group action and replace it by the somewhat weaker invariance with respect to the action of a Lie algebra $\mathfrak{g}$. Secondly, we make the conditions given in [12] more precise and we show that, given the existence of a classical momentum mapping, the existence of a quantum momentum mapping relies on two cohomological conditions on the formal series $\Omega \in \nu Z^2_{\mathrm{dR}}(M)[[\nu]]$ used to construct the $\mathfrak{g}$-invariant star product.

The paper is organized as follows. In Section 2, we collect some notations and give a very short review of Fedosov's construction, and we prove some technical details enabling us to describe explicitly all derivations of Fedosov star products $*$. In Section 3, we consider an arbitrary symplectic vector field X on M and we give necessary and sufficient conditions for X to be (via the Lie derivative) a derivation of $*$. Furthermore, we specify additional conditions guaranteeing that this derivation is even quasi-inner. In Section 4, we recall some definitions from [18] and we apply our results from Section 3 to give criteria for the invariance of $*$ with respect to some Lie algebra $\mathfrak{g}$. Finally, assuming the $\mathfrak{g}$-action to be Hamiltonian and the Hamiltonian to be equivariant with respect to the coadjoint action of $\mathfrak{g}$, we find necessary and sufficient conditions for the existence of a quantum momentum mapping.

2. Preliminaries

In this section we shall briefly recall the essentials of Fedosov's construction of star products on a symplectic manifold (M, ω). As we assume the reader to be familiar with this construction we shall restrict to the very minimum to introduce our notation. (For more details we refer the reader to [7,8] and [15, Section 2], where we even used the same notation.) Defining

$$\mathcal{W} \otimes \Lambda := \left(\mathsf{X}_{s=0}^{\infty} \Gamma^{\infty} \left(\bigvee^{s} T^{*}M \otimes \bigwedge T^{*}M \right) \right) [[\nu]], \tag{1}$$

it is obvious that $\mathcal{W} \otimes \Lambda$ becomes in a natural way an associative, super-commutative algebra and the product is denoted by $\mu(a \otimes b) = ab$ for $a, b \in \mathcal{W} \otimes \Lambda$. (By $\mathcal{W} \otimes \Lambda^k$ we denote the elements of anti-symmetric degree k and set $\mathcal{W} := \mathcal{W} \otimes \Lambda^0$.) Besides this pointwise product the Poisson tensor Λ corresponding to ω gives rise to another associative product $\circ$ on $\mathcal{W} \otimes \Lambda$ by

$$a \circ b = \mu \circ \exp(\tfrac{1}{2} \nu \Lambda^{ij} i_{\mathrm{s}}(\partial_i) \otimes i_{\mathrm{s}}(\partial_j))(a \otimes b), \tag{2}$$

which is a deformation of μ. Here $i_{\mathrm{s}}(Y)$ denotes the symmetric insertion of a vector field $Y \in \Gamma^{\infty}(TM)$ and similarly $i_{\mathrm{a}}(Y)$ shall be used to denote the anti-symmetric insertion of a vector field. We set $\mathrm{ad}(a)b := [a, b]$ where the latter denotes the $\deg_{\mathrm{a}}$-graded super-commutator with respect to $\circ$. Denoting the obvious degree-maps by $\deg_{\mathrm{s}}$, $\deg_{\mathrm{a}}$ and $\deg_{\nu} = \nu\partial_{\nu}$ one observes that they all are derivations with respect to μ but $\deg_{\mathrm{s}}$ and $\deg_{\nu}$ fail to be derivations with respect to $\circ$. Instead $\mathrm{Deg} := \deg_{\mathrm{s}} + 2\deg_{\nu}$ is a derivation of $\circ$ and hence $(\mathcal{W} \otimes \Lambda, \circ)$ is formally Deg-graded and the corresponding degree is referred to as the total degree. Sometimes we write $\mathcal{W}_k \otimes \Lambda$ to denote the elements of total degree $\geq k$.

In local coordinates we define the differential $\delta := (1 \otimes \mathrm{d}x^i) i_{\mathrm{s}}(\partial_i)$ which satisfies $\delta^2 = 0$ and is a super-derivation of $\circ$. Moreover, there is a homotopy operator δ^{-1} satisfying $\delta\delta^{-1} + \delta^{-1}\delta + \sigma = \mathrm{id}$ where $\sigma : \mathcal{W} \otimes \Lambda \to \mathcal{C}^{\infty}(M)[[\nu]]$ denotes the projection onto the part of symmetric and anti-symmetric degree 0 and $\delta^{-1} a := (1/(k+l))(\mathrm{d}x^i \otimes 1) i_{\mathrm{a}}(\partial_i) a$ for $\deg_{\mathrm{s}} a = ka$, $\deg_{\mathrm{a}} a = la$ with $k + l \neq 0$ and $\delta^{-1} a := 0$ else. From a torsion free symplectic connection ∇ on M we obtain a derivation $\nabla := (1 \otimes \mathrm{d}x^i)\nabla_{\partial_i}$ of $\circ$ that satisfies the following identities: $[\delta, \nabla] = 0$, $\nabla^2 = -(1/\nu)\,\mathrm{ad}(R)$, where $R := (1/4)\omega_{it} R^t_{jkl}\, \mathrm{d}x^i \vee \mathrm{d}x^j \otimes \mathrm{d}x^k \wedge \mathrm{d}x^l \in \mathcal{W} \otimes \Lambda^2$ involves the curvature of the connection. Moreover we have $\delta R = 0 = \nabla R$ by the Bianchi identities.

Now remember the following facts which are just restatements of Fedosov's original theorems in [7, Theorems 3.2 and 3.3] resp. [8, Theorem 5.3.3].

For all $\Omega \in \nu Z^2_{\mathrm{dR}}(M)[[\nu]]$ and all $s \in \mathcal{W}_3$ with $\sigma(s) = 0$ there exists a unique element $r \in \mathcal{W}_2 \otimes \Lambda^1$ such that

$$\delta r = \nabla r - \frac{1}{\nu} r \circ r + R + 1 \otimes \Omega \quad \text{and} \quad \delta^{-1} r = s. \tag{3}$$

Moreover r satisfies the formula

$$r = \delta s + \delta^{-1} \left(\nabla r - \frac{1}{\nu} r \circ r + R + 1 \otimes \Omega \right) \tag{4}$$

from which r can be determined recursively. In this case the Fedosov derivation

$$\mathcal{D} := -\delta + \nabla - \frac{1}{\nu} \mathrm{ad}(r) \tag{5}$$

is a super-derivation of anti-symmetric degree 1 and has square zero: $\mathcal{D}^2 = 0$. Furthermore observe that the $\mathcal{D}$-cohomology on elements a with positive anti-symmetric degree is trivial since one has the following homotopy formula $\mathcal{D}\mathcal{D}^{-1}a + \mathcal{D}^{-1}\mathcal{D}a = a$, where $\mathcal{D}^{-1}a := -\delta^{-1}(\mathrm{id} - [\delta^{-1}, \nabla - (1/\nu)\,\mathrm{ad}(r)])^{-1}a$ (cf. [8, Theorem 5.2.5]).

Then for any $f \in \mathcal{C}^\infty(M)[[\nu]]$ there exists a unique element $\tau(f) \in \ker(\mathcal{D}) \cap \mathcal{W}$ such that $\sigma(\tau(f)) = f$ and $\tau : \mathcal{C}^\infty(M)[[\nu]] \to \ker(\mathcal{D}) \cap \mathcal{W}$ is $\mathbb{C}[[\nu]]$-linear and τ is referred to as the Fedosov–Taylor series corresponding to $\mathcal{D}$. In addition $\tau(f)$ can be obtained recursively for $f \in \mathcal{C}^\infty(M)$ from

$$\tau(f) = f + \delta^{-1} \left(\nabla \tau(f) - \frac{1}{\nu} \mathrm{ad}(r) \tau(f) \right). \tag{6}$$

Using $\mathcal{D}^{-1}$ one can also write $\tau(f) = f - \mathcal{D}^{-1}(1 \otimes \mathrm{d}f)$. Since $\mathcal{D}$ as constructed above is a $\circ$-super-derivation $\ker(\mathcal{D}) \cap \mathcal{W}$ is a $\circ$-sub-algebra and a new associative product $*$ for $\mathcal{C}^\infty(M)[[\nu]]$, which turns out to be a star product, is defined by pull-back of $\circ$ via τ.

Observe that in (3) we allowed for an arbitrary element $s \in \mathcal{W}$ with $\sigma(s) = 0$ that contains no terms of total degree lower than 3 as normalization condition for r, i.e. $\delta^{-1}r = s$ instead of the usual equation $\delta^{-1}r = 0$. In the following we shall refer to the associative product $*$ defined above as the Fedosov star product (corresponding to (∇, Ω, s)).

Now we shall give a very convenient description of all derivations of the star product $*$ that will prove very useful for our further considerations. To this end we consider appropriate fibrewise quasi-inner derivations of the shape

$$\mathrm{D}_h = -\frac{1}{\nu} \mathrm{ad}(h), \tag{7}$$

where $h \in \mathcal{W}$ and without loss of generality we assume $\sigma(h) = 0$. Our aim is to define $\mathbb{C}[[\nu]]$-linear derivations of $*$ by $\mathcal{C}^\infty(M)[[\nu]] \ni f \mapsto \sigma(\mathrm{D}_h \tau(f))$ but for an arbitrary element $h \in \mathcal{W}$ with $\sigma(h) = 0$ this mapping fails to be a derivation as D_h does not map elements of $\ker(\mathcal{D}) \cap \mathcal{W}$ to elements of $\ker(\mathcal{D}) \cap \mathcal{W}$. In order to achieve this one must have that $\mathcal{D}$ and D_h super-commute. As $\mathcal{D}$ is a $\mathbb{C}[[\nu]]$-linear $\circ$-super-derivation we obviously have

$$[\mathcal{D}, \mathrm{D}_h] = -\frac{1}{\nu} \mathrm{ad}(\mathcal{D}h)$$

and hence obviously $\mathcal{D}h$ must be central, i.e. $\mathcal{D}h$ has to be of the shape $1 \otimes A$ with $A \in \Gamma^\infty(T^*M)[[\nu]]$ to have $[\mathcal{D}, \mathrm{D}_h] = 0$. From $\mathcal{D}^2 = 0$ we get that the necessary condition for the solvability of the equation $\mathcal{D}h = 1 \otimes A$ is the closedness of A since $\mathcal{D}(1 \otimes A) = 1 \otimes \mathrm{d}A$. But as the $\mathcal{D}$-cohomology is trivial on elements with positive anti-symmetric degree this condition is also sufficient for the solvability of the equation $\mathcal{D}h = 1 \otimes A$ and we get the following statement.

Lemma 2.1.

(i) *For all formal series $A \in \Gamma^\infty(T^*M)[[\nu]]$ of closed one-forms on M there is a uniquely determined element $h_A \in \mathcal{W}$ such that $\mathcal{D}h_A = 1 \otimes A$ and $\sigma(h_A) = 0$. Moreover h_A is explicitly given by*

$$h_A = \mathcal{D}^{-1}(1 \otimes A). \tag{8}$$

(ii) *For all $A \in Z^1_{\mathrm{dR}}(M)[[\nu]]$ the mapping $\mathsf{D}_A : \mathcal{C}^\infty(M)[[\nu]] \to \mathcal{C}^\infty(M)[[\nu]]$, where*

$$\mathsf{D}_A f := \sigma(\mathrm{D}_{h_A}\tau(f)) = \sigma\left(-\frac{1}{\nu}\mathrm{ad}(h_A)\tau(f)\right) \quad \forall f \in \mathcal{C}^\infty(M)[[\nu]] \tag{9}$$

defines a $\mathbb{C}[[\nu]]$-linear derivation of $*$ and hence this construction yields a mapping $Z^1_{\mathrm{dR}}(M)[[\nu]] \ni A \mapsto \mathsf{D}_A \in \mathrm{Der}_{\mathbb{C}[[\nu]]}(\mathcal{C}^\infty(M)[[\nu]], *)$.

Proof. The fact that $h_A = \mathcal{D}^{-1}(1 \otimes A)$ satisfies $\mathcal{D}h_A = 1 \otimes A$ is obvious from the homotopy formula for $\mathcal{D}$ and the closedness of A. In addition we have $\sigma(h_A) = 0$ since $\mathcal{D}^{-1}$ raises the symmetric degree at least by 1. For the uniqueness of h_A let $\tilde{h}_A$ be another solution of the equations above, then we obviously have $\mathcal{D}(h_A - \tilde{h}_A) = 0$ and hence $h_A - \tilde{h}_A = \tau(\varphi)$ for some $\varphi \in \mathcal{C}^\infty(M)[[\nu]]$. Applying σ to this equation one gets $\varphi = 0$, since $\sigma(h_A) = \sigma(\tilde{h}_A) = 0$ and $\sigma(\tau(\varphi)) = \varphi$, and hence $h_A = \tilde{h}_A$ proving that h_A is uniquely determined by the above equations. For the proof of (ii) we just observe that the equation $[\mathcal{D}, \mathrm{D}_{h_A}] = 0$ which is fulfilled according to (i) implies that $\mathrm{D}_{h_A}\tau(f) = \tau(\mathsf{D}_A f)$ for all $f \in \mathcal{C}^\infty(M)[[\nu]]$. Using this equation and the obvious fact that D_{h_A} is a derivation of $\circ$ it is straightforward to see using the very definition of $*$ that D_A as defined above is a derivation of $*$. The $\mathbb{C}[[\nu]]$-linearity of D_A is also evident from the $\mathbb{C}[[\nu]]$-linearity of τ. □

Furthermore we are now in the position to show that one even obtains all $\mathbb{C}[[\nu]]$-linear derivations of $*$ by varying A in the derivations D_A constructed above.

Proposition 2.2. *The mapping*

$$Z^1_{\mathrm{dR}}(M)[[\nu]] \ni A \mapsto \mathsf{D}_A \in \mathrm{Der}_{\mathbb{C}[[\nu]]}(\mathcal{C}^\infty(M)[[\nu]], *)$$

defined in Lemma 2.1 is a bijection. Moreover, we have that $\mathsf{D}_{\mathrm{d}f}$ is a quasi-inner derivation for all $f \in \mathcal{C}^\infty(M)[[\nu]]$, i.e. $\mathsf{D}_{\mathrm{d}f} = (1/\nu)\,\mathrm{ad}_*(f)$ and the induced mapping $[A] \mapsto [\mathsf{D}_A]$ from $H^1_{\mathrm{dR}}(M)[[\nu]] \cong Z^1_{\mathrm{dR}}(M)[[\nu]]/B^1_{\mathrm{dR}}(M)[[\nu]]$ to $\mathrm{Der}_{\mathbb{C}[[\nu]]}(\mathcal{C}^\infty(M)[[\nu]], *)/\mathrm{Der}^{\mathrm{qi}}_{\mathbb{C}[[\nu]]}(\mathcal{C}^\infty(M)[[\nu]], *)$ the space of $\mathbb{C}[[\nu]]$-linear derivations of $*$ modulo the quasi-inner derivations, also is bijective.

Proof. First we prove the injectivity of the mapping $A \mapsto \mathsf{D}_A$. To this end let $\mathsf{D}_A = \mathsf{D}_{A'}$ then we get from $\mathrm{D}_{h_A}\tau(f) = \tau(\mathsf{D}_A f)$ and from the analogous equation for A' that $\mathrm{ad}(h_A - h_{A'})\tau(f) = 0$ for all $f \in \mathcal{C}^\infty(M)[[\nu]]$ and hence $h_A - h_{A'}$ must be central (since it commutes with all Fedosov–Taylor series), i.e. we have $h_A - h_{A'} = g_{A,A'} \in \mathcal{C}^\infty(M)[[\nu]]$. But with $\sigma(h_A) = \sigma(h_{A'}) = 0$ this implies $g_{A,A'} = 0$ and hence $h_A = h_{A'}$ such that we get $1 \otimes A = \mathcal{D}h_A = \mathcal{D}h_{A'} = 1 \otimes A'$ proving the injectivity. For the surjectivity we start with an arbitrary derivation D of $*$ and we wish to find closed one-forms A_i such that $\mathsf{D} = \sum_{i=0}^\infty \nu^i \mathsf{D}_{A_i}$ inductively. Assume that we have found such one-forms for $0 \leq i \leq k-1$ such that $\mathsf{D}' = \mathsf{D} - \sum_{i=0}^{k-1} \nu^i \mathsf{D}_{A_i}$ – which obviously is again a derivation of $*$ – is of the shape $\mathsf{D}' = \sum_{i=k}^\infty \nu^i \mathsf{D}'_i$. The kth order in ν of the equation $\mathsf{D}'(f*g) = (\mathsf{D}'f)*g + f*(\mathsf{D}'g)$ for $f, g \in \mathcal{C}^\infty(M)$ yields that D'_k is a vector field $X_k \in \Gamma^\infty(TM)$. Considering the anti-symmetric part of $\mathsf{D}'(f * g) = (\mathsf{D}'f) * g + f * (\mathsf{D}'g)$ at order $k+1$ of ν we get that this vector field is symplectic, i.e. $\mathcal{L}_{X_k}\omega = 0$ and because of the Cartan formula $A_k := -i_{X_k}\omega$ defines a closed one-form on M. Considering the derivation D_{A_k} it is a straightforward computation using the explicit construction above to show that $\mathsf{D}_{A_k} f = X_k(f) + \mathrm{O}(\nu)$ for all $f \in \mathcal{C}^\infty(M)$. But then $\mathsf{D}' - \nu^k \mathsf{D}_{A_k}$ is again a derivation of $*$ that starts in order $k+1$ of ν and hence the surjectivity follows by induction. The fact that $\mathsf{D}_{\mathrm{d}f} = (1/\nu)\,\mathrm{ad}_*(f)$ for all $f \in \mathcal{C}^\infty(M)[[\nu]]$ is obvious from the observation that $\tau(f) = f - \mathcal{D}^{-1}(1 \otimes \mathrm{d}f)$ and the obvious fact that $\mathrm{ad}(f) = 0$. From the above, the well-definedness of the mapping $[A] \mapsto [\mathsf{D}_A]$ follows and the bijectivity is a direct consequence of the bijectivity of the mapping $A \mapsto \mathsf{D}_A$. □

Remark 2.3. Actually it is well known that for an arbitrary star product $\star$ on a symplectic manifold the space of $\mathbb{C}[[\nu]]$-linear derivations is in bijection with $Z^1_{\mathrm{dR}}(M)[[\nu]]$ and that the quotient space of these derivations modulo the quasi-inner derivations is in bijection with $H^1_{\mathrm{dR}}(M)[[\nu]]$ (cf. [4, Theorem 4.2], observe that the proof given above is just an adaption of the idea of the general proof to our special situation) but the remarkable thing about Fedosov star products is that these bijections can be explicitly expressed in terms of $\mathcal{D}$ and $\mathcal{D}^{-1}$ in a very lucid way which will be useful in the following.

To conclude this section we shall remove some redundancy in the description of the star products $*$ by (∇, Ω, s). This will ease the more detailed analysis in the following section. To this end we shall recall some well-known facts about symplectic torsion free connections on (M, ω). Given two such connections say ∇ and ∇' it is obvious that $S^{\nabla-\nabla'}(X, Y) := \nabla_X Y - \nabla'_X Y$ where $X, Y \in \Gamma^\infty(TM)$ defines a symmetric tensor field $S^{\nabla-\nabla'} \in \Gamma^\infty(\bigvee^2 T^*M \otimes TM)$ on M. Defining $\sigma^{\nabla-\nabla'}(X, Y, Z) := \omega(S^{\nabla-\nabla'}(X, Y), Z)$ it is easy to see that $\sigma^{\nabla-\nabla'} \in \Gamma^\infty(\bigvee^3 T^*M)$ is a totally symmetric tensor field. Vice versa given an arbitrary element $\sigma \in \Gamma^\infty(\bigvee^3 T^*M)$ and a symplectic torsion free connection ∇ and defining $S^\sigma \in \Gamma^\infty(\bigvee^2 T^*M \otimes TM)$ by $\sigma(X, Y, Z) = \omega(S^\sigma(X, Y), Z)$ then ∇^σ defined by $\nabla^\sigma_X Y := \nabla_X Y - S^\sigma(X, Y)$ again is a symplectic torsion free connection and all such connections can be obtained this way by varying σ. Using these relations we shall compare the corresponding mappings ∇ and ∇' on $\mathcal{W} \otimes \Lambda$ in the following lemma.

Lemma 2.4. *With the notations from above we have*

$$\nabla - \nabla' = -(\mathrm{d}x^j \otimes \mathrm{d}x^i)i_{\mathrm{s}}(S^{\nabla-\nabla'}(\partial_i, \partial_j)) = \frac{1}{\nu}\mathrm{ad}(T^{\nabla-\nabla'}), \tag{10}$$

where the tensor field $T^{\nabla-\nabla'} \in \Gamma^\infty(\bigvee^2 T^*M \otimes T^*M) \subseteq \mathcal{W} \otimes \Lambda^1$ *is defined by* $T^{\nabla-\nabla'}(Z, Y; X) := \sigma^{\nabla-\nabla'}(X, Y, Z) = \omega(S^{\nabla-\nabla'}(X, Y), Z)$. *Moreover* $T^{\nabla-\nabla'}$ *satisfies the equations*

$$\delta T^{\nabla-\nabla'} = 0 \quad \text{and} \quad \begin{aligned} \nabla T^{\nabla-\nabla'} &= R' - R + \frac{1}{\nu} T^{\nabla-\nabla'} \circ T^{\nabla-\nabla'}, \\ \nabla' T^{\nabla-\nabla'} &= R' - R - \frac{1}{\nu} T^{\nabla-\nabla'} \circ T^{\nabla-\nabla'}, \end{aligned} \tag{11}$$

where $R = (1/4)\omega_{it} R^t_{jkl}\, \mathrm{d}x^i \vee \mathrm{d}x^j \otimes \mathrm{d}x^k \wedge \mathrm{d}x^l$ *and* $R' = (1/4)\omega_{it} R'^t_{jkl}\, \mathrm{d}x^i \vee \mathrm{d}x^j \otimes \mathrm{d}x^k \wedge \mathrm{d}x^l$ *denote the corresponding elements of* $\mathcal{W} \otimes \Lambda^2$ *that are built from the curvature tensors of* ∇ *and* ∇'.

Proof. The proof of (10) is a straightforward computation using the very definitions from above. The first identity in (11) directly follows from (10) and $[\delta, \nabla] = [\delta, \nabla'] = 0$. The other identities in (11) are also easily obtained squaring Eq. (10). □

Now we are in the position to compare two Fedosov derivations $\mathcal{D}$ and $\mathcal{D}'$ resp. the induced star products $*$ and $*'$ obtained from (∇, Ω, s) and (∇', Ω', s').

Proposition 2.5. *The Fedosov derivations* $\mathcal{D}$ *and* $\mathcal{D}'$ *coincide if and only if* $T^{\nabla-\nabla'} - r + r' = 1 \otimes \vartheta$ *where* $\vartheta \in \nu\Gamma^\infty(T^*M)[[\nu]]$ *which is equivalent to*

$$\sigma^{\nabla-\nabla'} \otimes 1 - s + s' = \vartheta \otimes 1 \quad \text{and} \quad \Omega - \Omega' = \mathrm{d}\vartheta. \tag{12}$$

Proof. Writing down the definitions of $\mathcal{D}$ and $\mathcal{D}'$ using Eq. (10) the first equivalence is obvious since $T^{\nabla-\nabla'} - r + r'$ is central in $(\mathcal{W} \otimes \Lambda, \circ)$ if and only if $\mathcal{D} = \mathcal{D}'$. For the proof of the second equivalence first assume that we have $T^{\nabla-\nabla'} - r + r' = 1 \otimes \vartheta$. Applying δ^{-1} to this equation and using the normalization condition on r and r' we obtain the first equation in (12) since $\delta^{-1} T^{\nabla-\nabla'} = \sigma^{\nabla-\nabla'} \otimes 1$. In order to obtain the second equation in (12) we apply δ to $T^{\nabla-\nabla'} - r + r' = 1 \otimes \vartheta$ and a straightforward computation using the equations for r and r' together with the identities from (11) yields the stated result. To prove that the converse is also true assume that the equations in (12) are satisfied and define $B := r - r' - T^{\nabla-\nabla'} + 1 \otimes \vartheta \in \mathcal{W}_2 \otimes \Lambda^1$. Then again a straightforward computation yields that B satisfies $\mathcal{D}B = -(1/\nu) B \circ B$ and $\delta^{-1} B = 0$ such that the homotopy formula for δ together with $\sigma(B) = 0$ implies that B is the unique fixed point of the mapping $\mathcal{W}_2 \otimes \Lambda^1 \ni a \mapsto \delta^{-1}(\nabla a - (1/\nu)\,\mathrm{ad}(r) a + (1/\nu) a \circ a) \in \mathcal{W}_2 \otimes \Lambda^1$ (cf. [16, Appendix B] for the application of Banach's fixed point theorem in this framework). But 0 trivially is a fixed point of this mapping and hence uniqueness implies that $B = 0$ proving the other direction of the second stated equivalence. □

As an important direct consequence of this proposition we get the following deduction.

Deduction 2.6. *For every Fedosov star product* $*$ *obtained from* (∇, Ω, s) *with* $s \in \mathcal{W}_3$ *there is a connection* ∇', *a formal series* Ω' *of closed two-forms and an element* $s' \in \mathcal{W}_4$ *without* *terms of symmetric degree* 1 *such that the star product obtained from* (∇', Ω', s') *coincides with* $*$, *and hence we may without loss of generality restrict to such normalization conditions when varying the connection and the formal series of closed two-forms arbitrarily*.

Proof. We write $s = s' + \sigma \otimes 1 - \vartheta \otimes 1$ and the preceding proposition states that $\mathcal{D}$ coincides with $\mathcal{D}'$ (and hence the corresponding star products coincide) where $\mathcal{D}'$ is obtained from $\Omega' = \Omega - \mathrm{d}\vartheta$ and $\nabla' = \nabla - (1/\nu)\,\mathrm{ad}(\delta(\sigma \otimes 1))$. □

3. Symplectic vector fields as derivations of $*$

Throughout this and the following section let $*$ denote the Fedosov star product obtained from (∇, Ω, s) as in Section 2 where in view of Deduction 2.6 we may assume that $s \in \mathcal{W}_4$ contains no part of symmetric degree 1. Furthermore $X \in \Gamma^\infty(TM)$ shall always denote a symplectic vector field on (M, ω) and the space of all these vector fields shall be denoted by $\Gamma^\infty_{\mathrm{symp}}(TM) := \{Y \in \Gamma^\infty(TM) | \mathcal{L}_Y \omega = 0\}$. It seems to be folklore and actually is not very hard to prove that the conditions $[\mathcal{L}_X, \nabla] = 0$, $\mathcal{L}_X \Omega = 0 = \mathcal{L}_X s$ are sufficient to guarantee that the Lie derivative with respect to X is a derivation of $*$. Besides providing a very simple proof of this fact, our aim in this section is to prove that the converse is also true, i.e. the conditions given above are also necessary to have that X defines a derivation of $*$. Moreover, we find an additional cohomological condition involving ω, Ω and X that is equivalent to $\mathcal{L}_X$ being even a quasi-inner derivation.

As an important tool we need the deformed Cartan formula (cf. [15, Appendix A]) that relates the Lie derivative with respect to a symplectic vector field X with the Fedosov derivation $\mathcal{D}$.

Lemma 3.1. *For all* $X \in \Gamma^\infty_{\mathrm{symp}}(TM)$ *the Lie derivative* $\mathcal{L}_X$ *can be expressed in the following manner*:

$$\mathcal{L}_X = \mathcal{D} i_{\mathrm{a}}(X) + i_{\mathrm{a}}(X)\mathcal{D} - \frac{1}{\nu}\mathrm{ad}\left(\theta_X \otimes 1 + \frac{1}{2} D\theta_X \otimes 1 - i_{\mathrm{a}}(X) r\right), \tag{13}$$

where $D := \mathrm{d}x^i \vee \nabla_{\partial_i}$ *denotes the operator of symmetric covariant derivation and the closed one-form* θ_X *is defined by* $\theta_X := i_X \omega$.

Proof. Since the Lie derivative is a local operator it suffices to prove the above identity over any contractible open subset U of M. But as X is symplectic it is locally Hamiltonian, i.e. over U there is a function $f \in \mathcal{C}^\infty(U)$ such that $X|_U = X_f$ resp. $\mathrm{d}f = \theta_X|_U$. For Hamiltonian vector fields the Cartan formula as above was proved in [15, Proposition 5] and hence Eq. (13) is valid for all symplectic vector fields $X \in \Gamma^\infty_{\mathrm{symp}}(TM)$. □

As an immediate consequence of this lemma we obtain:

Lemma 3.2. *For* $X \in \Gamma^\infty_{\mathrm{symp}}(TM)$ *the Lie derivative* $\mathcal{L}_X$ *is a derivation with respect to* $\circ$. *In addition we have* $[\delta, \mathcal{L}_X] = [\delta^{-1}, \mathcal{L}_X] = 0$.

Proof. The first statement of the lemma is obvious from Eq. (13) and the commutation relations follow from the fact that $\mathcal{L}_X$ is compatible with contractions and preserves the symmetric and the anti-symmetric degree. □

After these rather technical preparations we get the following proposition.

Proposition 3.3. *Let $X \in \Gamma^\infty_{\mathrm{symp}}(TM)$ then $\mathcal{L}_X$ is a derivation of $*$ if and only if $[\mathcal{L}_X, \mathcal{D}] = 0$ which is equivalent to the existence of a formal series $A_X \in Z^1_{\mathrm{dR}}(M)[[\nu]]$ of closed one-forms such that $\mathcal{D}(\theta_X \otimes 1 + (1/2)D\theta_X \otimes 1 - i_{\mathrm{a}}(X)r) = 1 \otimes A_X$.*

Proof. First let us assume that $[\mathcal{L}_X, \mathcal{D}] = 0$ then the obvious equation $\mathcal{L}_X \circ \sigma = \sigma \circ \mathcal{L}_X$ implies that $\mathcal{L}_X \tau(f) = \tau(\mathcal{L}_X f)$ for all $f \in \mathcal{C}^\infty(M)[[\nu]]$. But with this equation and the fact that $\mathcal{L}_X$ is a derivation of $\circ$ it is straightforward to prove that $\mathcal{L}_X$ is a derivation of $*$. Assuming that $\mathcal{L}_X$ is a derivation of $*$ Proposition 2.2 implies that there is a formal series A_X of closed one-forms on M such that $\mathcal{L}_X f = \sigma(-(1/\nu)\,\mathrm{ad}(\mathcal{D}^{-1}(1 \otimes A_X))\tau(f))$ but on the other hand the deformed Cartan formula yields $\mathcal{L}_X f = \sigma(-(1/\nu)\,\mathrm{ad}(\theta_X \otimes 1 + (1/2)D\theta_X \otimes 1 - i_{\mathrm{a}}(X)r)\tau(f))$ and hence $\mathcal{D}^{-1}(1 \otimes A_X) - (\theta_X \otimes 1 + (1/2)D\theta_X \otimes 1 - i_{\mathrm{a}}(X)r)$ has to be central, i.e. a formal function. Observing that $\mathcal{D}^{-1}$ raises the symmetric degree at least by 1 and that r contains no part of symmetric degree 0 which is due to the special shape of the normalization condition this implies $\mathcal{D}^{-1}(1 \otimes A_X) = (\theta_X \otimes 1 + (1/2)D\theta_X \otimes 1 - i_{\mathrm{a}}(X)r)$. Applying $\mathcal{D}$ to this equation and using the homotopy formula for $\mathcal{D}$ together with the fact that A_X is closed we get $\mathcal{D}(\theta_X \otimes 1 + (1/2)D\theta_X \otimes 1 - i_{\mathrm{a}}(X)r) = 1 \otimes A_X$. Assuming finally that this equation is fulfilled, the deformed Cartan formula together with $\mathcal{D}^2 = 0$ obviously implies $[\mathcal{L}_X, \mathcal{D}] = 0$ since $1 \otimes A_X$ is central and hence the proposition is proved. □

We shall now go on by analysing the condition

$$\mathcal{D}(\theta_X \otimes 1 + \tfrac{1}{2}D\theta_X \otimes 1 - i_{\mathrm{a}}(X)r) = 1 \otimes A_X, \quad \text{where } \mathrm{d}A_X = 0 \tag{14}$$

in more detail in order to find out whether it gives rise to conditions on (∇, Ω, s) and X.

Lemma 3.4. *For all symplectic vector fields $X \in \Gamma^\infty_{\mathrm{symp}}(TM)$ we have*

$$\begin{aligned}\mathcal{D}(\theta_X \otimes 1 &+ \tfrac{1}{2}D\theta_X \otimes 1 - i_{\mathrm{a}}(X)r)\\ &= -1 \otimes \theta_X + \nabla(\tfrac{1}{2}D\theta_X \otimes 1) - \mathcal{L}_X r - i_{\mathrm{a}}(X)R - 1 \otimes i_X\Omega.\end{aligned} \tag{15}$$

Proof. The proof of this equation is a straightforward computation using the equation that is solved by r and the deformed Cartan formula (13) once again. □

Next we shall need some detailed formulas that describe $[\nabla, \mathcal{L}_X]$ in order to simplify the result of the above lemma. The proofs of the following two lemmas are just slight variations of the proofs of [15, Lemmas 3 and 4].

Lemma 3.5. *For all $X \in \Gamma^\infty_{\mathrm{symp}}(TM)$ the mapping $[\nabla, \mathcal{L}_X]$ enjoys the following properties*:

(i) *In local coordinates one has*

$$[\nabla, \mathcal{L}_X] = (\mathrm{d}x^j \otimes \mathrm{d}x^i) i_{\mathrm{s}}((\mathcal{L}_X \nabla)_{\partial_i} \partial_j) = (\mathrm{d}x^j \otimes \mathrm{d}x^i) i_{\mathrm{s}}(S_X(\partial_i, \partial_j)), \tag{16}$$

where the tensor field $S_X \in \Gamma^\infty(T^*M \otimes T^*M \otimes TM)$ is defined by

$$\begin{aligned} S_X(\partial_i, \partial_j) &= (\mathcal{L}_X \nabla)_{\partial_i} \partial_j := \mathcal{L}_X \nabla_{\partial_i} \partial_j - \nabla_{\partial_i} \mathcal{L}_X \partial_j - \nabla_{\mathcal{L}_X \partial_i} \partial_j \\ &= R(X, \partial_i) \partial_j + \nabla^{(2)}_{(\partial_i, \partial_j)} X. \end{aligned} \tag{17}$$

(ii) *S_X as defined above is symmetric, i.e. $S_X \in \Gamma^\infty(\bigvee^2 T^*M \otimes TM)$.*
(iii) *For all $U, V, W \in \Gamma^\infty(TM)$ we have $\omega(W, S_X(U, V)) = -\omega(S_X(U, W), V)$.*

Now S_X naturally gives rise to an element $T_X \in \Gamma^\infty(\bigvee^2 T^*M \otimes T^*M)$ of $\mathcal{W} \otimes \Lambda^1$ of symmetric degree 2 and anti-symmetric degree 1 by

$$T_X(W, U; V) := \omega(W, S_X(V, U)) \tag{18}$$

and we obtain:

Lemma 3.6. *The tensor field T_X as defined in (18) satisfies the following equations*:

(i) $(1/\nu)\,\mathrm{ad}(T_X) = [\nabla, \mathcal{L}_X]$;
(ii) $T_X = i_{\mathrm{a}}(X) R - \nabla((1/2) D\theta_X \otimes 1)$;
(iii) $\delta T_X = 0$ *and* $\nabla T_X = \mathcal{L}_X R$.

From the preceding lemma we find that the result of Lemma 3.4 simplifies to

$$\mathcal{D}(\theta_X \otimes 1 + \tfrac{1}{2} D\theta_X \otimes 1 - i_{\mathrm{a}}(X) r) = -1 \otimes \theta_X - T_X - \mathcal{L}_X r - 1 \otimes i_X \Omega. \tag{19}$$

Finally we have to find equations that determine $\mathcal{L}_X r$ in order to analyse Eq. (14).

Lemma 3.7. *Let X denote a symplectic vector field then $\mathcal{L}_X r$ satisfies the equations*

$$\begin{aligned} &\delta \mathcal{L}_X r = \nabla \mathcal{L}_X r - \frac{1}{\nu} \mathrm{ad}(r) \mathcal{L}_X r - \frac{1}{\nu} \mathrm{ad}(T_X) r + \mathcal{L}_X R + 1 \otimes \mathrm{d} i_X \Omega \\ &\textit{and} \quad \delta^{-1} \mathcal{L}_X r = \mathcal{L}_X s \end{aligned} \tag{20}$$

from which $\mathcal{L}_X r$ is uniquely determined and can be computed recursively from

$$\mathcal{L}_X r = \delta \mathcal{L}_X s + \delta^{-1} \left(\nabla \mathcal{L}_X r - \frac{1}{\nu} \mathrm{ad}(r) \mathcal{L}_X r - \frac{1}{\nu} \mathrm{ad}(T_X) r + \mathcal{L}_X R + 1 \otimes \mathrm{d} i_X \Omega \right).$$

Proof. For the proof of (20) one just has to apply $\mathcal{L}_X$ to the equations that determine r and to use the commutation relations of the involved mappings. From these equations it is straightforward to find the recursion formula for $\mathcal{L}_X r$ using the homotopy formula for δ. Using statement (iii) of Lemma 3.6 the argument for the uniqueness of the solution of these equations is completely analogous to the one used to prove the uniqueness of r and hence we leave it to the reader. □

After all these preparations we are in the position to formulate the main results of this section.

Theorem 3.8. *Let X be a symplectic vector field and let $*$ be the Fedosov star product corresponding to (∇, Ω, s), where $s \in \mathcal{W}_4$ contains no part of symmetric degree 1. Then, $\mathcal{L}_X$ is a derivation of $*$ if and only if $T_X = 0$, $\mathcal{L}_X\Omega = 0$ and $\mathcal{L}_X s = 0$, i.e. if and only if X is affine with respect to ∇ and both s and Ω are invariant with respect to X.*

Proof. First let $T_X = 0 = \mathcal{L}_X\Omega = \mathcal{L}_X s$ then we have $\mathcal{L}_X R = \nabla T_X = 0$ and $\mathrm{d}i_X\Omega = 0$ and hence $\mathcal{L}_X r = \delta^{-1}(\nabla\mathcal{L}_X r - (1/\nu)\,\mathrm{ad}(r)\mathcal{L}_X r)$. But this implies $\mathcal{L}_X r = 0$ and then obviously $[\mathcal{D}, \mathcal{L}_X] = (1/\nu)\,\mathrm{ad}(T_X + \mathcal{L}_X r) = 0$ such that Proposition 3.3 implies that $\mathcal{L}_X$ is a derivation of $*$. To prove the converse we again use Proposition 3.3 which says that in case $\mathcal{L}_X$ is a derivation of $*$ there is a formal series A_X of closed one-forms on M such that $\mathcal{D}(\theta_X \otimes 1 + (1/2)D\theta_X \otimes 1 - i_{\mathrm{a}}(X)r) = 1 \otimes A_X$. Together with Eq. (19) this yields $\mathcal{L}_X r = -(1 \otimes (\theta_X + A_X + i_X\Omega) + T_X)$. Applying δ^{-1} to this equation and using the second equation in (20) we get

$$\mathcal{L}_X s = -(\theta_X + A_X + i_X\Omega) \otimes 1 - \delta^{-1}T_X.$$

Now s and hence $\mathcal{L}_X s$ is in $\mathcal{W}_4$ and has no part of symmetric degree 1 such that this equation implies $\mathcal{L}_X s = 0$, $\theta_X + A_X + i_X\Omega = 0$ and $\delta^{-1}T_X = 0$. Since θ_X and A_X are closed the second of these equations implies $0 = \mathrm{d}i_X\Omega = \mathcal{L}_X\Omega$ and using the homotopy formula for δ together with $\delta T_X = 0$ the last equation yields $T_X = 0$ which is equivalent to X being affine with respect to ∇ according to Lemmas 3.5 and 3.6. Finally one can insert the above expression for $\mathcal{L}_X r$ into the first equation in (20) which turns out to be satisfied identically, which is just a check for consistency. □

In the case $s = 0$ a different proof of the statement of the above theorem was given in [10] (cf. also [11, Lemma 6.1]). This proof uses the facts that every Fedosov star product is natural in the sense of [11, Definition 2.1] and that every natural star product uniquely determines a torsion free symplectic connection, that for a Fedosov star product coincides with the connection used to construct it, whereas our proof remains within the framework of Fedosov's construction and makes no use of other additional results than those presented in this section.

Finally we can give an additional condition for $\mathcal{L}_X$ to be even a quasi-inner derivation of $*$ which is originally due to Gutt [10] (cf. also [11, Theorem 6.2]).

Proposition 3.9. *Let X be a symplectic vector field such that $\mathcal{L}_X$ is a derivation of $*$ then $\mathcal{L}_X$ is even quasi-inner if and only if there is a formal function $f \in \mathcal{C}^\infty(M)[[\nu]]$ such that*

$$\mathrm{d}f = \theta_X + i_X\Omega = i_X(\omega + \Omega) \tag{21}$$

and then $\mathcal{L}_X = \mathcal{L}_{X_{f_0}} = -(1/\nu)\,\mathrm{ad}_*(f)$, where we have written $f = f_0 + f_+$ with $f_0 \in \mathcal{C}^\infty(M)$ and $f_+ \in \nu\mathcal{C}^\infty(M)[[\nu]]$.

Proof. From Eq. (13) it is obvious that $\mathcal{L}_X$ is quasi-inner if and only if there is a formal function $f \in \mathcal{C}^\infty(M)[[\nu]]$ such that $\tau(f) = f + \theta_X \otimes 1 + (1/2)D\theta_X \otimes 1 - i_{\mathrm{a}}(X)r$ but using

Eq. (19) together with $T_X = 0$, $\mathcal{L}_X r = 0$ and $\mathcal{D} f = 1 \otimes \mathrm{d} f$ this is equivalent to (21). In fact the necessary condition for the solvability of this equation is fulfilled since $i_X \Omega$ is closed according to Theorem 3.8 and θ_X is closed as X is symplectic. Moreover, observe that the zeroth order in ν of (21) just means that X is Hamiltonian with Hamiltonian function f_0 and hence the second statement of the proposition is immediate. □

The proof of the fact that the existence of a formal function f that satisfies (21) is sufficient for $\mathcal{L}_X$ to be quasi-inner is originally due to Kravchenko [13, Proposition 4.3].

4. $\mathfrak{g}$-Invariant star products $*$ and quantum momentum mappings

In this section we shall use the results of Theorem 3.8 to find necessary and sufficient conditions for the star product $*$ to be invariant with respect to a Lie algebra action. Furthermore Proposition 3.9 gives criteria for the existence of a quantum Hamiltonian and with some little more effort we shall find a last condition which is necessary and sufficient for this quantum Hamiltonian to define a quantum momentum mapping for $*$.

First let us recall some definitions from [18]. Let us consider a finite dimensional real or complex Lie algebra $\mathfrak{g}$ and let $X_\cdot : \mathfrak{g} \to \Gamma^\infty_{\mathrm{symp}}(TM) : \xi \mapsto X_\xi$ denote a Lie algebra anti-homomorphism, i.e. $[X_\xi, X_\eta] = -X_{[\xi,\eta]}$ for all $\xi, \eta \in \mathfrak{g}$. Then obviously $\varrho(\xi) f := -\mathcal{L}_{X_\xi} f$ defines a Lie algebra action of $\mathfrak{g}$ on $\mathcal{C}^\infty(M)$ that naturally extends to a Lie algebra action on $\mathcal{C}^\infty(M)[[\nu]]$.

Definition 4.1. With the notations from above a star product $\star$ is called $\mathfrak{g}$-invariant in case $\varrho(\xi)$ is a derivation of $\star$ for all $\xi \in \mathfrak{g}$.

From Theorem 3.8 we obviously get the following deduction.

Deduction 4.2. *The Fedosov star product $*$ constructed from (∇, Ω, s), where $s \in \mathcal{W}_4$ contains no part of symmetric degree 1, is $\mathfrak{g}$-invariant if and only if X_ξ is affine with respect to ∇ for all $\xi \in \mathfrak{g}$, i.e. $[\nabla, \mathcal{L}_{X_\xi}] = 0\, \forall\, \xi \in \mathfrak{g}$ and Ω and s are invariant with respect to X_ξ for all $\xi \in \mathfrak{g}$, i.e. $\mathrm{d} i_{X_\xi} \Omega = \mathcal{L}_{X_\xi} \Omega = 0 = \mathcal{L}_{X_\xi} s\, \forall\, \xi \in \mathfrak{g}$.*

Let us introduce some notation. Considering some complex vector space V endowed with a representation $\pi : \mathfrak{g} \to \mathrm{Hom}(V, V)$ of the Lie algebra $\mathfrak{g}$ in V we denote the space of V-valued k-multilinear alternating forms on $\mathfrak{g}$ by $C^k(\mathfrak{g}, V)$ and the corresponding Chevalley–Eilenberg differential shall be denoted by $\delta_\pi : C^\bullet(\mathfrak{g}, V) \to C^{\bullet+1}(\mathfrak{g}, V)$. Moreover the spaces of the corresponding cocycles, coboundaries, and the corresponding cohomology spaces shall be denoted by $Z^k_\pi(\mathfrak{g}, V)$, $B^k_\pi(\mathfrak{g}, V)$, and $H^k_\pi(\mathfrak{g}, V)$.

Now the Lie algebra action ϱ is called Hamiltonian if and only if there is an element $J_0 \in C^1(\mathfrak{g}, \mathcal{C}^\infty(M))$ such that $X_{J_0(\xi)} = X_\xi$ for all $\xi \in \mathfrak{g}$, i.e. $i_{X_\xi} \omega = \mathrm{d} J_0(\xi)$. In this case $\varrho(\xi) \cdot = \{J_0(\xi), \cdot\}$ and J_0 is said to be a Hamiltonian for the action ϱ. (For applications in physics where typically $\mathfrak{g}$ is the real Lie algebra corresponding to a Lie group that acts on M by symplectomorphisms and where the generating vector fields X_ξ are real-valued the Hamiltonian J_0 is assumed to be real-valued, too.) In case J_0 is equivariant with respect to the coadjoint representation of $\mathfrak{g}$, i.e. $\{J_0(\xi), J_0(\eta)\} = J_0([\xi, \eta])$ for all $\xi, \eta \in \mathfrak{g}$ one calls J_0 a classical momentum mapping.

Definition 4.3. Let $\star$ be a $\mathfrak{g}$-invariant star product, then $J = J_0 + J_+ \in C^1(\mathfrak{g}, \mathcal{C}^\infty(M))[[\nu]]$ with $J_0 \in C^1(\mathfrak{g}, \mathcal{C}^\infty(M))$ and $J_+ \in \nu C^1(\mathfrak{g}, \mathcal{C}^\infty(M))[[\nu]]$ is called a quantum Hamiltonian for the action ϱ in case

$$\varrho(\xi) = \frac{1}{\nu}\mathrm{ad}_\star(J(\xi)) \quad \text{for all } \xi \in \mathfrak{g}. \tag{22}$$

J is called a quantum momentum mapping if in addition

$$\frac{1}{\nu}(J(\xi) \star J(\eta) - J(\eta) \star J(\xi)) = J([\xi, \eta]) \tag{23}$$

for all $\xi, \eta \in \mathfrak{g}$.

Observe that the zeroth order in ν of (22) is equivalent to J_0 being a Hamiltonian for ϱ and that the zeroth order in ν of (23) just means equivariance of this classical Hamiltonian with respect to the coadjoint action of $\mathfrak{g}$ or equivalently that J_0 is a classical momentum mapping. For Fedosov star products the fact that J_0 has to be a classical Hamiltonian for ϱ can also be seen directly from Proposition 3.9 as we have the following deduction.

Deduction 4.4. *A $\mathfrak{g}$-invariant Fedosov star product for (M, ω) obtained from (∇, Ω, s) admits a quantum Hamiltonian if and only if there is an element $J \in C^1(\mathfrak{g}, \mathcal{C}^\infty(M))[[\nu]]$ such that*

$$\mathrm{d}J(\xi) = i_{X_\xi}(\omega + \Omega) \quad \forall\, \xi \in \mathfrak{g}, \tag{24}$$

i.e. if and only if $[i_{X_\xi}(\omega + \Omega)] = [0]\,\forall\, \xi \in \mathfrak{g}$. Moreover, from Eq. (24) J is determined (in case it exists) up to elements in $C^1(\mathfrak{g}, \mathbb{C})[[\nu]]$.

Remark 4.5. Observe that the condition $H^1_{\mathrm{dR}}(M) = 0$ is obviously sufficient for the existence of a quantum Hamiltonian for an arbitrary $\mathfrak{g}$-invariant star product $\star$ since then any $\mathbb{C}[[\nu]]$-linear derivation of $\star$ is quasi-inner. But for $\mathfrak{g}$-invariant Fedosov star products $*$ the condition for the existence of a quantum Hamiltonian is much weaker and more precise since only the cohomology classes of very special closed one-forms have to vanish and not the complete cohomology.

Now recall the definition of a strongly invariant star product from [1].

Definition 4.6. Let J_0 be a classical momentum mapping for the action ϱ. Then a $\mathfrak{g}$-invariant star product is called strongly invariant if and only if $J = J_0$ defines a quantum Hamiltonian for this action.

Observe that the notion of strong invariance does not depend on the chosen classical momentum mapping since every classical momentum mapping is of the form $J_0 + b$ with $b \in Z^1_0(\mathfrak{g}, \mathbb{C})$ and hence every classical momentum mapping defines a quantum Hamiltonian for ϱ in case J_0 does. Moreover, in the case of a strongly invariant star product $\star$ every classical momentum mapping J_0 obviously yields a quantum momentum mapping $J = J_0$ since $(1/\nu)\,\mathrm{ad}_\star(J_0(\xi))J_0(\eta) = \{J_0(\xi), J_0(\eta)\} = J_0([\xi,\eta])$ for all $\xi,\eta \in \mathfrak{g}$. As an immediate corollary of Deduction 4.4 we find the following statement.

Corollary 4.7. *Let J_0 be a classical momentum mapping for the action ϱ. Then a $\mathfrak{g}$-invariant Fedosov star product $*$ obtained from (∇, Ω, s) is strongly invariant if and only if*

$$i_{X_\xi}\Omega = 0 \quad \text{for all}\ \ \xi \in \mathfrak{g}. \tag{25}$$

In this case every classical momentum mapping defines a quantum momentum mapping for $*$.

Proof. According to Deduction 4.4 a classical momentum mapping J_0 defines a quantum Hamiltonian for $*$ if and only if $\mathrm{d}J_0(\xi) = i_{X_\xi}(\omega+\Omega)$ for all $\xi \in \mathfrak{g}$ but since $\mathrm{d}J_0(\xi) = i_{X_\xi}\omega$ this is equivalent to Eq. (25). □

Returning to the general case our next aim is to give a further condition involving ω, Ω and $X.$ which in addition guarantees that a quantum Hamiltonian J is in fact a quantum momentum mapping.

Proposition 4.8. *Let J be a quantum Hamiltonian for the Fedosov star product $*$ then $\lambda \in C^2(\mathfrak{g}, \mathcal{C}^\infty(M))[[\nu]]$ defined by*

$$\lambda(\xi,\eta) := \frac{1}{\nu}(J(\xi) * J(\eta) - J(\eta) * J(\xi)) - J([\xi,\eta]) \tag{26}$$

lies in $C^2(\mathfrak{g},\mathbb{C})[[\nu]]$ and is an element of $Z_0^2(\mathfrak{g},\mathbb{C})[[\nu]]$ which is explicitly given by

$$\lambda(\xi,\eta) = (\omega+\Omega)(X_\xi, X_\eta) - J([\xi,\eta]) \tag{27}$$

and the cohomology class $[\lambda] \in H_0^2(\mathfrak{g},\mathbb{C})[[\nu]]$ does not depend on the choice of J. Moreover quantum momentum mappings exist if and only if $[\lambda] = [0] \in H_0^2(\mathfrak{g},\mathbb{C})[[\nu]]$ and for every $a \in C^1(\mathfrak{g},\mathbb{C})[[\nu]]$ such that $\delta_0 a = \lambda$ the element $J^a := J - a \in C^1(\mathfrak{g}, \mathcal{C}^\infty(M))[[\nu]]$ is a quantum momentum mapping for $*$. Finally, the quantum momentum mapping (if it exists) is unique up to elements in $Z_0^1(\mathfrak{g},\mathbb{C})[[\nu]]$, and hence we have uniqueness if and only if $H_0^1(\mathfrak{g},\mathbb{C}) = 0$.

Proof. In fact all the statements of the proposition except for the explicit shape of λ hold for any $\mathfrak{g}$-invariant star product $\star$ according to [18, Proposition 6.3] and are straightforward to prove. It thus remains to prove (27) but this follows from the following computation using Eq. (24):

$$\begin{aligned}\lambda(\xi,\eta) + J([\xi,\eta]) &= \frac{1}{\nu}\mathrm{ad}_*(J(\xi))J(\eta) = -\mathcal{L}_{X_\xi}J(\eta) = -i_{X_\xi}\,\mathrm{d}J(\eta)\\ &= -i_{X_\xi}i_{X_\eta}(\omega+\Omega) = (\omega+\Omega)(X_\xi, X_\eta).\end{aligned}$$

□

Again, for Fedosov star products the second condition for the existence of a quantum momentum mapping can be formulated more precisely than in the general case since the cocycle λ whose cohomology class has to vanish to get a quantum momentum mapping can be expressed explicitly in terms of ω, Ω and $X_\cdot$. Obviously, supposing the existence of a classical Hamiltonian for ϱ the zeroth order of this condition is equivalent to the existence of a classical momentum mapping.

Let us consider the important example of a semi-simple Lie algebra $\mathfrak{g}$ in more detail.

Example 4.9. In case $\mathfrak{g}$ is semi-simple it is well known that one has the following properties: $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g} (\Rightarrow H^1_0(\mathfrak{g},\mathbb{C}) = 0)$ and $H^2_0(\mathfrak{g},\mathbb{C}) = 0$. But then $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ implies writing $\xi = \sum_{k\in I}[\zeta^{(k)},\eta^{(k)}]$ (the sum ranges over a finite index set I) with $\zeta^{(k)},\eta^{(k)} \in \mathfrak{g}$ and using the invariance of $\omega+\Omega$ with respect to $X_{\zeta^{(k)}}$ and $X_{\eta^{(k)}}$ that

$$
\begin{aligned}
i_{X_\xi}(\omega+\Omega) &= -\sum_{k\in I} i_{[X_{\zeta^{(k)}},X_{\eta^{(k)}}]}(\omega+\Omega)\\
&= -\sum_{k\in I}\mathcal{L}_{X_{\zeta^{(k)}}} i_{X_{\eta^{(k)}}}(\omega+\Omega) = \mathrm{d}\left(\sum_{k\in I}(\omega+\Omega)(X_{\zeta^{(k)}},X_{\eta^{(k)}})\right)
\end{aligned}
$$

and hence for all $\xi\in\mathfrak{g}$ there is a $J(\xi)\in\mathcal{C}^\infty(M)[[\nu]]$ such that $\mathrm{d}J(\xi) = i_{X_\xi}(\omega+\Omega)$. Moreover, one can achieve that $J\in C^1(\mathfrak{g},\mathcal{C}^\infty(M))[[\nu]]$ implying that J defines a quantum Hamiltonian for $*$ (e.g. fix a basis $\{e_i\}_{1\le i\le \dim(\mathfrak{g})}$ of $\mathfrak{g}$, write $e_i = \sum_{k\in I_i}[\zeta_i^{(k)},\eta_i^{(k)}]$, define $J(e_i) := \sum_{k\in I_i}(\omega+\Omega)(X_{\zeta_i^{(k)}},X_{\eta_i^{(k)}})$ such that $\mathrm{d}J(e_i) = i_{X_{e_i}}(\omega+\Omega)$ holds according to the above computation and extend J to $\mathfrak{g}$ by linearity yielding $J\in C^1(\mathfrak{g},\mathcal{C}^\infty(M))[[\nu]]$ with $\mathrm{d}J(\xi) = i_{X_\xi}(\omega+\Omega)\,\forall\,\xi\in\mathfrak{g}$). This observation together with the statements of Proposition 4.8 and $H^1_0(\mathfrak{g},\mathbb{C}) = H^2_0(\mathfrak{g},\mathbb{C}) = 0$ implies that in this case there is a unique quantum momentum mapping for every $\mathfrak{g}$-invariant Fedosov star product.

Returning to the case of an arbitrary Lie algebra $\mathfrak{g}$ we also have the following corollary.

Corollary 4.10. *Let $*$ be a $\mathfrak{g}$-invariant Fedosov star product and assume that there is a classical momentum mapping J_0 for the action ϱ, then a quantum momentum mapping J exists if and only if there is an element $J_+\in\nu C^1(\mathfrak{g},\mathcal{C}^\infty(M))[[\nu]]$ such that*

$$
i_{X_\xi}\Omega = \mathrm{d}J_+(\xi) \quad \text{and} \quad \Omega(X_\xi,X_\eta) = (\delta_\varrho J_+)(\xi,\eta) \quad \forall\,\xi,\eta\in\mathfrak{g}, \tag{28}
$$

and these equations determine J_+ up to elements of $\nu Z^1_0(\mathfrak{g},\mathbb{C})[[\nu]]$.

Proof. Assuming the existence of a classical momentum mapping it is obvious that (24) and the equation $\lambda(\xi,\eta) = 0$ for all $\xi,\eta\in\mathfrak{g}$ reduce to $i_{X_\xi}\Omega = \mathrm{d}J_+(\xi)$ and $J_+([\xi,\eta]) = \Omega(X_\xi,X_\eta)$ and it is straightforward to see that these two equations are equivalent to (28). The statement about the ambiguity of J_+ is obvious from Proposition 4.8. □

Observe that the condition for the existence of a quantum momentum mapping for $\mathfrak{g}$-invariant Fedosov star products given in the above corollary does not depend on the chosen classical momentum mapping but only on Ω and $X_\cdot$. Moreover, our result shows that the answer to the question whether existence of a classical momentum mapping implies the existence of a quantum momentum mapping posed in [18] in general is no if one allows for star products whose characteristic class is different from $(1/\nu)[\omega]$ since the conditions above involve the two-form Ω that determines this class (cf. [15]) and that has to be different from zero in this case. One can even construct very simple examples where Ω is even exact and hence the characteristic class is equal to $(1/\nu)[\omega]$ but nevertheless there exists no quantum momentum mapping.

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